

Analytical Evaluation of Double Boxes¹

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Abstract

Recent results on the analytical evaluation of double-box Feynman integrals and the corresponding methods of evaluation are briefly reviewed.

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Introduction.

Feynman diagrams with four external lines contribute to many important physical quantities. They are rather complicated mathematical objects because they are functions of multiple variables: internal masses, two independent Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and squares of external momenta. The most complicated diagrams are the planar double box and non-planar (crossed) double box shown in Fig. 1.

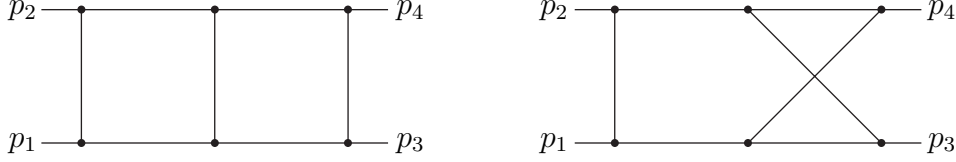


Figure 1: The planar and non-planar double boxes

Almost all available analytical results correspond to the massless diagrams. Ironically, the first result for the massless double boxes was obtained in the most complicated case, where all the four external legs are off-shell, i.e. $p_i^2 \neq 0$ for all $i = 1, 2, 3, 4$. This is the elegant analytical result for the scalar planar master double box, i.e. for all powers of propagators equal to one, obtained in [1]:

$$\frac{(i\pi^2)^2}{s^2 t} C(p_1^2 p_4^2, p_2^2 p_3^2, st) \quad (1)$$

where

$$\begin{aligned} C(x_1, x_2, x_3) = & \frac{1}{\lambda} (6 [\text{Li}_4(-\rho x) + \text{Li}_4(-\rho y)] \\ & + 3 \ln \frac{y}{x} [\text{Li}_3(-\rho x) - \text{Li}_3(-\rho y)] + \frac{1}{2} \ln^2 \frac{y}{x} [\text{Li}_2(-\rho x) + \text{Li}_2(-\rho y)] \\ & + \frac{1}{4} \ln^2(\rho x) \ln^2(\rho y) + \frac{\pi^2}{2} \ln(\rho x) \ln(\rho y) + \frac{\pi^2}{12} \ln^2 \frac{y}{x} + \frac{7\pi^4}{60}) , \end{aligned} \quad (2)$$

$$\lambda(x, y) = \sqrt{(1 - x - y)^2 - 4xy} , \quad \rho(x, y) = \frac{2}{1 - x - y + \lambda(x, y)} , \quad (3)$$

$x = x_1/x_3$, $y = x_2/x_3$, and $\text{Li}_n(z)$ is the polylogarithm [2].

This diagram is convergent both in the ultraviolet and infrared sense. (For example, if one puts a dot on some line, an infrared divergence appears so that a regularization is needed.) To obtain this result, the authors have exploited the technique of Feynman parameters and Mellin–Barnes (MB) representation. However, no

other results for the pure off-shell double boxes have been derived up to now so that the above result stays unique in the pure off-shell category.

For massless double-box diagrams with at least one leg on the mass shell, i.e. $p_i^2 = 0$, infrared and collinear divergences appear, so that one introduces a regularization which is usually chosen to be dimensional [3], with the space-time dimension d as a regularization parameter. One hardly believes that a regularized double-box diagram can be analytically evaluated for the general value of the regularization parameter $\epsilon = (4 - d)/2$, and the evaluation is usually performed in a Laurent expansion in ϵ , typically, up to a finite part.

The problem of the evaluation of Feynman integrals associated with a given graph according to some Feynman rules is usually decomposed into two parts: reduction of general Feynman integrals of this class to so-called master integrals (which cannot be simplified further) and the evaluation of these master integrals. A standard tool to solve the first part of this problem is the method of integration by parts (IBP) [4] when one writes down identities obtained by putting to zero various integrals of derivatives of the general integrand connected with the given graph and tries to solve a resulting system of equations to obtain recurrence relations that express Feynman integrals with general integer powers of the propagators through the master integrals.

The purpose of this brief review is to characterize the status of the analytical evaluation of double-box diagrams and describe the corresponding techniques. We shall first deal with the pure on-shell case, i.e. $p_i^2 = 0$, $i = 1, 2, 3, 4$, where this problem was completely solved during last three years, i.e. all the master integrals were calculated in expansion in ϵ and a reduction procedure was developed both in the planar and non-planar case [5, 6, 7, 8, 9, 10, 11].

We also describe a method based on alpha (or Feynman) parameters and Mellin–Barnes (MB) representation used to calculate the master integrals, starting with a much simpler example of the on-shell box diagram. It turns out that the calculation of the basic master double-box diagram [5] based on a fivefold MB representation was far from being optimal. As it has been shown in [11] it is possible to go through a fourfold MB representation. We shall describe here how this can be done.

Then we turn to an intermediate massless case where one of the external legs is on-shell and the other three legs are on-shell, i.e. $p_1^2 = q^2 \neq 0$, $p_i^2 = 0$, $i = 2, 3, 4$ and list results of last two years where the problem of the evaluation was solved in this situation [12, 13]. We then consider the massive on-shell case and describe the only available and recently obtained result for the master planar double box [14]. We conclude with a discussion of perspectives of the analytical evaluation of the double-box diagrams.

The pure on-shell case: a box.

To illustrate the method of MB representation let us evaluate the massless scalar on-shell box diagram of Fig. 2. The Feynman integral can be written as

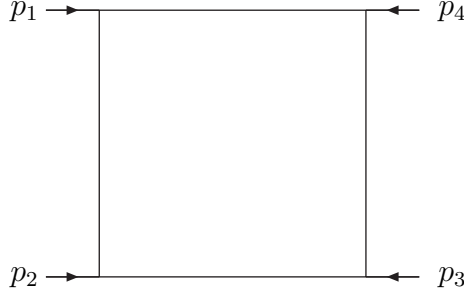


Figure 2: Box diagram.

$$F(s, t; d) = \int \frac{d^d k}{(k^2 + 2p_1 \cdot k)(k^2 - 2p_2 \cdot k)k^2(k + p_1 + p_2)^2} , \quad (4)$$

where usual prescriptions $k^2 = k^2 + i0$, etc. are implied. Using alpha or Feynman parameters we arrive at the following three-dimensional parametric integral:

$$F(s, t; d) = i\pi^{d/2}\Gamma(2 + \epsilon) \int_0^1 \int_0^1 \int_0^1 d\xi_1 d\xi_2 d\eta \eta(1 - \eta) \times \left[-s\xi_1(1 - \xi_1)(1 - \eta)^2 - t\xi_2(1 - \xi_2)\eta^2 - i0 \right]^{-2-\epsilon} . \quad (5)$$

Let us now use the MB representation

$$\frac{1}{(X + Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z) , \quad (6)$$

where the contour of integration is chosen in the standard way: the poles with a $\Gamma(\dots + z)$ dependence are to the left of the contour and the poles with a $\Gamma(\dots - z)$ dependence are to the right of it. Representation (6) is applied to the factor with square brackets in (5). As a result, the two terms in the square brackets in (5) raised to the power $-2 - \epsilon$ are replaced by a product of some powers of these terms (at the cost of introducing an extra integration), and, after evaluating parametric integrals in terms of gamma functions, we obtain

$$F(s, t; d) = \frac{i\pi^{d/2}}{(-s)^{2+\epsilon}\Gamma(-2\epsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \left(\frac{t}{s} \right)^z \times \Gamma(2 + \epsilon + z) \Gamma(1 + z)^2 \Gamma(-1 - \epsilon - z)^2 \Gamma(-z) , \quad (7)$$

where the same prescription for dealing with poles is implied, i.e. the poles of $\Gamma(2 + \epsilon + z)\Gamma(1 + z)^2$ are to the left of the integration contour and the poles of $\Gamma(-1 - \epsilon - z)^2\Gamma(-z)$ are to the right of it. In the case $\text{Re } \epsilon < 0$, we can choose this contour to be a straight line parallel to the imaginary axis, while in the case $\text{Re } \epsilon > 0$, a more complicated contour has to be chosen. Anyway, if we put $\epsilon = 0$, the integral becomes

ill-defined because the first pole with a $\Gamma(\dots + z)$ dependence, i.e. at $z = -1$, and the first pole with a $\Gamma(\dots - z)$ dependence, i.e. at $z = -1 - \epsilon$, glue together and there is no space to satisfy the prescriptions for the contours.

This observation shows how the poles of the Feynman integral in ϵ are generated, from the point of view of MB integrals. The next step is to pick up the pole in ϵ by taking a residue, e.g. at the pole $z = -1 - \epsilon$ (with the minus sign, of course) and shifting the integration contour across this point. A resulting integral can be taken over the line at $-1 < \text{Re } z < 0$. There is no gluing in this integral so that the integral can be safely expanded in a Taylor series in ϵ . Every term of this expansion can be integrated by closing the integration contour to the right, taking residues at the points $z = 0, 1, 2, \dots$, and summing up a resulting series. Taking into account the terms up to ϵ^1 and combining them with the value of the above residue we arrive at the following result:

$$F(s, t; d) = \frac{i\pi^{d/2} e^{-\gamma_E \epsilon}}{st} \left\{ \frac{4}{\epsilon^2} - [\ln(-s) + \ln(-t)] \frac{2}{\epsilon} + 2 \ln(-s) \ln(-t) - \frac{4\pi^2}{3} \right. \\ \left. + \epsilon \left[\text{Li}_3\left(-\frac{t}{s}\right) - 2 \ln \frac{t}{s} \text{Li}_2\left(-\frac{t}{s}\right) - \left(\ln^2 \frac{t}{s} + \pi^2\right) \ln\left(1 + \frac{t}{s}\right) \right] \right\} + O(\epsilon^2), \quad (8)$$

where γ_E is the Euler constant.

The pure on-shell case: the basic master diagram.

Let us now consider the general on-shell planar double box diagram of Fig. 1, i.e. with a general irreducible numerator and powers of propagators. We choose

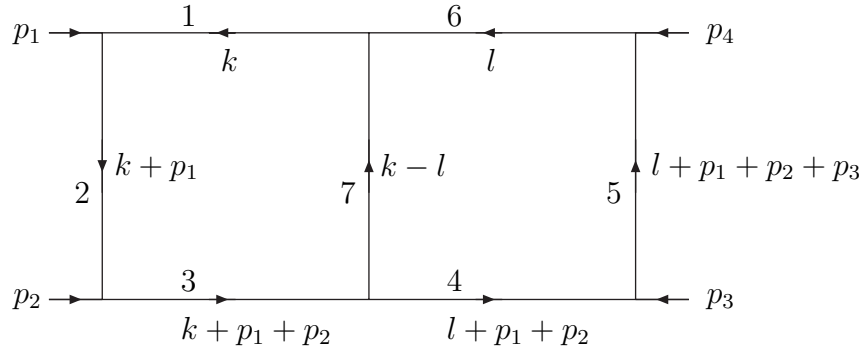


Figure 3: Planar double box diagram.

this irreducible numerator and the routing of the external momenta as in [11] — see Fig. 3. For convenience, we consider the factor with $(k + p_1 + p_2 + p_3)^2$ corresponding to the irreducible numerator as an extra propagator but, really, we are interested only in the non-positive integer values of a_8 . This general double box Feynman integral takes the form

$$K(a_1, \dots, a_8; s, t; \epsilon) = \int \int \frac{d^d k d^d l}{(k^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2]^{a_3}}$$

$$\times \frac{[(k + p_1 + p_2 + p_3)^2]^{-a_8}}{[(l + p_1 + p_2)^2]^{a_4} [(l + p_1 + p_2 + p_3)^2]^{a_5} (l^2)^{a_6} [(k - l)^2]^{a_7}}, \quad (9)$$

where k and l are respectively loop momenta of the left and the right box.

To resolve the singularity structure of Feynman integrals in ϵ it is very useful to apply the MB representation (6) that makes it possible to replace sums of terms raised to some power by their products in some powers, at the cost of introducing extra integrations. In [5, 7, 12] MB integrations were introduced directly in alpha/Feynman parametric integrals. It turns out more convenient to follow (as in [11]) the strategy of [1] and introduce, in a suitable way, MB integrations, first, after integration over one of the loop momenta, l , and complete this procedure after integration over the second loop momentum, k . After appropriate changes of variables we arrive at the following fourfold MB representation of (9) (see also [11]):

$$\begin{aligned} K(a_1, \dots, a_8; s, t; \epsilon) &= \frac{(i\pi^{d/2})^2 (-1)^a}{\prod_{j=2,4,5,6,7} \Gamma(a_j) \Gamma(4 - a_{4567} - 2\epsilon) (-s)^{a-4+2\epsilon}} \\ &\times \frac{1}{(2\pi i)^4} \int_{-i\infty}^{+i\infty} dw dz_2 dz_3 dz_4 \left(\frac{t}{s}\right)^w \frac{\Gamma(a_2 + w) \Gamma(-w) \Gamma(z_2 + z_4) \Gamma(z_3 + z_4)}{\Gamma(a_1 + z_3 + z_4) \Gamma(a_3 + z_2 + z_4)} \\ &\times \frac{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3) \Gamma(a_{1238} - 2 + \epsilon + z_4) \Gamma(a_7 + w - z_4)}{\Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - z_2 - z_3)} \\ &\times \frac{\Gamma(a_{4567} - 2 + \epsilon + w - z_4) \Gamma(a_8 - z_2 - z_3 - z_4) \Gamma(-w - z_2 - z_3 - z_4)}{\Gamma(4 - a_{1238} - 2\epsilon + w - z_4) \Gamma(a_8 - w - z_2 - z_3 - z_4)} \\ &\times \frac{\Gamma(a_5 + w + z_2 + z_3 + z_4) \Gamma(2 - a_{567} - \epsilon - w - z_2)}{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3)} \\ &\times \Gamma(2 - a_{457} - \epsilon - w - z_3) \Gamma(2 - a_{128} - \epsilon + z_2) \Gamma(2 - a_{238} - \epsilon + z_3) \\ &\times \Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - z_2 - z_3), \end{aligned} \quad (10)$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$, $a_{13} = a_1 + a_3$, etc., and integration contours are chosen in the standard way.

In the case of the master double box, we set $a_i = 1$ for $i = 1, 2, \dots, 7$ and $a_8 = 0$ and obtain

$$\begin{aligned} K^{(0)}(s, t; \epsilon) &\equiv K(1, \dots, 1, 0; s, t; \epsilon) \\ &= -\frac{(i\pi^{d/2})^2}{\Gamma(-2\epsilon) (-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^4} \int_{-i\infty}^{+i\infty} dw dz_2 dz_3 dz_4 \left(\frac{t}{s}\right)^w \frac{\Gamma(1 + w) \Gamma(-w)}{\Gamma(1 - 2\epsilon + w - z_4)} \\ &\times \frac{\Gamma(2 + \epsilon + w - z_4) \Gamma(-1 - \epsilon - w - z_2) \Gamma(-1 - \epsilon - w - z_3)}{\Gamma(1 + z_2 + z_4) \Gamma(1 + z_3 + z_4)} \\ &\times \Gamma(1 + w + z_2 + z_3 + z_4) \Gamma(1 + \epsilon + z_4) \Gamma(z_2 + z_4) \Gamma(z_3 + z_4) \\ &\times \Gamma(-\epsilon + z_2) \Gamma(-\epsilon + z_3) \Gamma(1 + w - z_4) \Gamma(-z_2 - z_3 - z_4). \end{aligned} \quad (11)$$

Observe that, because of the presence of the factor $\Gamma(-2\epsilon)$ in the denominator, we

are forced to take some residue in order to arrive at a non-zero result at $\epsilon = 0$, so that the integral is effectively threefold.

Of course, the resolution of singularities in ϵ in such a multi-dimensional MB integral is much more complicated than in the one-dimensional case. The poles in ϵ are not visible at once, at a first integration over one of the MB variables. However, the rule for finding a mechanism of the generation of poles is just a straightforward generalization of the rule used in the previous one-loop example, where we saw that the product of $\Gamma(-1 - \epsilon - z)\Gamma(1 + z)$ generated the pole of the type $\Gamma(-\epsilon)$ (this is nothing but the value of one of these gamma function at the pole of the other gamma function). Suppose we start from the integration over one of the variables, z . We analyze various products $\Gamma(a + z)\Gamma(b - z)$, where a and b depend on the rest of the variables, with the understanding that this integration generates a pole of the type $\Gamma(a + b)$. This means that any contour of one the next integrations should be chosen according to this dependence. We continue this analysis with various integrations at the second step, etc.

Here is an example of the procedure of generating poles in the integral (11). The product $\Gamma(-1 - \epsilon - w - z_2)\Gamma(-\epsilon + z_2)$ generates, due to the integration over z_2 , a pole of the type $\Gamma(-1 - 2\epsilon - w)$. Then the product of this gamma function with $\Gamma(1 + w)$ generates a pole of the type $\Gamma(2\epsilon)$ due to the integration in w .

After such preliminary analysis we conclude that the key gamma functions that are responsible for the generation of poles in ϵ are $\Gamma(-\epsilon + z_2)$, $\Gamma(-\epsilon + z_3)$ and $\Gamma(1 + w - z_4)$. This gives a hint for the construction of a complete procedure of the resolution of the singularities in ϵ , with the goal to decompose the given integral into pieces where the Laurent expansion of the integrand in ϵ becomes possible. One can proceed as follows.

We first take care of the gamma functions $\Gamma(-\epsilon + z_2)$ and $\Gamma(-\epsilon + z_3)$, i.e. take residues at $z_2 = \epsilon$ and $z_3 = \epsilon$ and shift contours across these poles. As a result, (11) is decomposed as $K = K_{11} + K_{10} + K_{01} + K_{00}$, where K_{11} corresponds to taking the two residues, K_{00} is defined by the same expression (11) but with both first poles of the selected two gamma functions treated in the opposite way, and the two intermediate contributions are defined by taking one of the residues and changing the nature of the first pole of the other gamma function.

The contribution K_{11} takes the form

$$\begin{aligned}
K_{11} = & -\frac{(i\pi^{d/2})^2}{\Gamma(-2\epsilon)(-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^4} \int_{-i\infty}^{+i\infty} dw dz_4 \left(\frac{t}{s}\right)^w \Gamma(1 + w) \Gamma(-1 - 2\epsilon - w)^2 \\
& \times \Gamma(-w) \Gamma(1 + w - z_4) \Gamma(2 + \epsilon + w - z_4) \Gamma(\epsilon + z_4)^2 \Gamma(-2\epsilon - z_4) \\
& \times \frac{\Gamma(1 + 2\epsilon + w + z_4)}{\Gamma(1 - 2\epsilon + w - z_4) \Gamma(1 + \epsilon + z_4)}. \tag{12}
\end{aligned}$$

The contributions K_{10} and K_{01} are equal to each other because of the symmetrical

dependence of the integrand on z_2 and z_3 . We have

$$\begin{aligned}
K_{01} = & -\frac{(i\pi^{d/2})^2}{\Gamma(-2\epsilon)(-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} dw dz_2 dz_4 \left(\frac{t}{s}\right)^w \Gamma(1+w)\Gamma(-1-2\epsilon-w) \\
& \times \Gamma(-w)\Gamma(-1-\epsilon-w-z_2)\Gamma(-\epsilon+z_2)\Gamma(1+w-z_4)\Gamma(2+\epsilon+w-z_4) \\
& \frac{\Gamma(\epsilon+z_4)\Gamma(z_2+z_4)\Gamma(-\epsilon-z_2-z_4)\Gamma(1+\epsilon+w+z_2+z_4)}{\Gamma(1-2\epsilon+w-z_4)\Gamma(1+z_2+z_4)}, \quad (13)
\end{aligned}$$

where the first pole of $\Gamma(-\epsilon+z_2)$ is of the opposite nature.

For all these contributions, further decompositions are necessary. One can proceed as follows.

(K_{11}) Take minus residue at $w = -1 - 2\epsilon$. The resulting one-dimensional integral is calculated by taking care of the first poles of $\Gamma(z_4)$ and $\Gamma(z_4 + \epsilon)$. One obtains either an explicit expression in gamma and ψ functions and a one-dimensional MB integral. In the integral, where the first pole of $\Gamma(-1 - 2\epsilon - w)$ is of the opposite nature, take care of the first pole of $\Gamma(z_4 + \epsilon)$. One obtains a one-dimensional MB integral over w which is calculated by expanding in ϵ . The integral, where the first poles of $\Gamma(-1 - 2\epsilon - w)$ and $\Gamma(z_4 + \epsilon)$ are of the opposite nature, does not contribute because one can expand it in ϵ and obtain a zero result due to $\Gamma(-2\epsilon)$ in the denominator.

(K_{01}) Take minus residue at $w = -1 - 2\epsilon$ and consecutively take care of the first poles of the gamma functions $\Gamma(z_4 + \epsilon)$, $\Gamma(z_2 + z_4)$ and $\Gamma(z_2 + z_4 - \epsilon)$ in the resulting integral. One obtains one-dimensional MB integrals and a two-dimensional integral in z_2 and z_4 which is calculated by use of the second Barnes lemma (see below). In the integral with the pole of $\Gamma(-1 - 2\epsilon - w)$ of the opposite nature, one takes care of the first pole of $\Gamma(z_2 + z_4)$, i.e. takes a residue at $z_4 = -z_2$ (and then takes care of the first pole of $\Gamma(-1 - \epsilon - w - z_2)$) and considers an integral with the opposite nature of this pole (with a zero contribution).

(K_{00}) Take care of the first poles of $\Gamma(-1 - \epsilon - w - z_2)$ and $\Gamma(-1 - \epsilon - w - z_3)$. The only non-zero contribution arises when taking both residues.

As a result we obtain either explicit expression in terms of gamma functions and their derivatives, or one-dimensional integrals over straight lines parallel to the imaginary axis of ratios of gamma functions which can be of two types: integrals over w or some z -variable. The integrals over w can be calculated by closing contour to the right, taking residues at the points $w = 0, 1, 2, \dots$ and summing up resulting series, with the help of the table of formulae presented in [15]. The one-dimensional MB integrals in z_i can be calculated with the help of the first and the second Barnes lemmas

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dw \Gamma(\lambda_1 + w)\Gamma(\lambda_2 + w)\Gamma(\lambda_3 - w)\Gamma(\lambda_4 - w) \\
& = \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}, \quad (14)
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dw \frac{\Gamma(\lambda_1 + w)\Gamma(\lambda_2 + w)\Gamma(\lambda_3 + w)\Gamma(\lambda_4 - w)\Gamma(\lambda_5 - w)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + w)} \\
&= \frac{\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_4)\Gamma(\lambda_3 + \lambda_4)\Gamma(\lambda_1 + \lambda_5)\Gamma(\lambda_2 + \lambda_5)\Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5)\Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}
\end{aligned} \tag{15}$$

and their corollaries. These are two typical examples of such corollaries:

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dw \frac{\Gamma(\lambda_1 + w)\Gamma(\lambda_2 + w)^2\Gamma(-\lambda_2 - w)\Gamma(\lambda_3 - w)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + w)} \\
&= \frac{\Gamma(\lambda_1 - \lambda_2)\Gamma(\lambda_2 + \lambda_3) [\psi'(\lambda_1 + \lambda_3) - \psi'(\lambda_2 + \lambda_3)]}{\Gamma(\lambda_1 + \lambda_3)},
\end{aligned} \tag{16}$$

where the nature of the pole at $w = -\lambda_2$ is determined by $\Gamma(\lambda_2 + w)$ while other poles are treated in the standard way, and

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} dw \Gamma(1+w)\Gamma(w)\Gamma(-w)\Gamma(-1-w)\psi(1+w)^2 \\
&= \frac{\gamma_E^2 \pi^2}{3} + 6\gamma_E \zeta(3) + \frac{\pi^4}{45}.
\end{aligned} \tag{17}$$

Here $\zeta(z)$ is the Riemann zeta function.

Collecting all the contributions, one reproduces the result of [5]:

$$K^{(0)}(s, t; \epsilon) = \frac{\left(i\pi^{d/2} e^{-\gamma_E \epsilon}\right)^2}{(-s)^{2+2\epsilon}(-t)} K(t/s; \epsilon), \tag{18}$$

where

$$\begin{aligned}
K(x, \epsilon) = & -\frac{4}{\epsilon^4} + \frac{5 \ln x}{\epsilon^3} - \left(2 \ln^2 x - \frac{5}{2} \pi^2\right) \frac{1}{\epsilon^2} \\
& - \left(\frac{2}{3} \ln^3 x + \frac{11}{2} \pi^2 \ln x - \frac{65}{3} \zeta(3)\right) \frac{1}{\epsilon} \\
& + \frac{4}{3} \ln^4 x + 6\pi^2 \ln^2 x - \frac{88}{3} \zeta(3) \ln x + \frac{29}{30} \pi^4 \\
& - \left[2 \text{Li}_3(-x) - 2 \ln x \text{Li}_2(-x) - (\ln^2 x + \pi^2) \ln(1+x)\right] \frac{2}{\epsilon} \\
& - 4[S_{2,2}(-x) - \ln x S_{1,2}(-x)] + 44 \text{Li}_4(-x) \\
& - 4[\ln(1+x) + 6 \ln x] \text{Li}_3(-x) \\
& + 2\left(\ln^2 x + 2 \ln x \ln(1+x) + \frac{10}{3} \pi^2\right) \text{Li}_2(-x) \\
& + (\ln^2 x + \pi^2) \ln^2(1+x) \\
& - \frac{2}{3} [4 \ln^3 x + 5\pi^2 \ln x - 6\zeta(3)] \ln(1+x) + O(\epsilon),
\end{aligned} \tag{19}$$

where, in addition to the polylogarithms, one meets also the generalized polylogarithms [16]

$$S_{a,b}(z) = \frac{(-1)^{a+b-1}}{(a-1)!b!} \int_0^1 \frac{\ln^{a-1}(t) \ln^b(1-zt)}{t} dt. \quad (20)$$

This result is in agreement with the leading behaviour in the (Regge) limit $t/s \rightarrow 0$ obtained by use of the strategy of expansion by regions [17, 18, 19]. It turns out that the (h-h), (1c-1c) and (2c-2c) contributions are the only non-zero contributions. (See [18, 19] for definitions of the hard (h) and collinear ((1c) and (2c)) regions and the corresponding contributions.) Keeping the two leading powers in x we have [6]

$$\begin{aligned} K(x, \epsilon) = & -\frac{4}{\epsilon^4} + \frac{5 \ln x}{\epsilon^3} - \left(2 \ln^2 x - \frac{5}{2} \pi^2\right) \frac{1}{\epsilon^2} \\ & - \left(\frac{2}{3} \ln^3 x + \frac{11}{2} \pi^2 \ln x - \frac{65}{3} \zeta(3)\right) \frac{1}{\epsilon} \\ & + \frac{4}{3} \ln^4 x + 6 \pi^2 \ln^2 x - \frac{88}{3} \zeta(3) \ln x + \frac{29}{30} \pi^4 \\ & + 2x \left(\frac{1}{\epsilon} (\ln^2 x - 2 \ln x + \pi^2 + 2)\right) \\ & - \frac{1}{3} \left\{4 \ln^3 x + 3 \ln^2 x + (5 \pi^2 - 36) \ln x + 2[33 + 5 \pi^2 - 3 \zeta(3)]\right\} \\ & + O(x^2 \ln^3 x, \epsilon). \end{aligned} \quad (21)$$

The pure on-shell case: the non-planar case and reduction to master diagrams.

The basic master non-planar on-shell massless double-box diagram were calculated in [7] by the same method of Feynman parameters and MB representation, with the only qualification that the initial integration contours were chosen, at appropriate values of ϵ , as straight lines parallel to imaginary axis. Then the procedure of taking residues and shifting contours was applied, with the requirement to keep this property.³ It turns out that it is natural to consider non-planar double boxes as functions of the three Mandelstam variables s, t and $u = (p_1 + p_4)^2$ not necessarily restricted by the physical condition $s + t + u = 0$ which does not simplify the result.

Reduction procedures for the evaluation of general double-box diagrams, with arbitrary numerators and integer powers of the propagators were developed in [6] in the planar case and in [9] in the non-planar case. Both reduction procedures were based on solving recurrence relations following from IBP [4]. In the non-planar case, the so-called Lorentz invariance (LI) identities were used, in addition to the IBP relations. In [6], the first of the two most complicated master integrals involved is $K^{(0)} \equiv K(1, \dots, 1, 0)$ considered above. As a second complicated master

³I think, this is a matter of taste what variant of resolution of the singularities in ϵ to apply. Anyway, I have confirmed the result of [7] using the strategy described above.

integral, the authors of [6] have chosen the diagram with a dot on the central line, i.e. $K(1, 1, 1, 1, 1, 2, 1, 0)$. As was pointed out later [21], in practical calculations one runs into a linear combination of these two master integrals with a coefficient $1/\epsilon$, so that a problem has arisen because the calculation of the master integrals in one more order in ϵ looked rather nasty. Two solutions of this problem have immediately appeared. In [10], the authors have calculated this very combination of the master integrals, while in [11] another choice of the master integrals has been made: instead of $K(1, 1, 1, 1, 1, 2, 1, 0)$, the authors have taken the integral $K(1, 1, 1, 1, 1, 1, 1, -1)$ as the second complicated master integral. This was a more successful choice because, according to the calculational experience, no negative powers of ϵ occur as coefficients at these two master integrals.

These analytical algorithms have been successfully applied to the evaluation of two-loop virtual corrections to various scattering processes [22] in the zero-mass approximation.

One leg off-shell.

In the case, where one of the external legs is on-shell, $p_1^2 \neq 0$, $p_i^2 = 0$, $i = 2, 3, 4$, the planar double box and one of two possible non-planar double-box diagram with all powers of propagators equal to one were analytically calculated in [12], as functions of the Mandelstam variables s and t and the non-zero external momentum squared p_1^2 . Explicit results were expressed through (generalized) polylogarithms, up to the fourth order, dependent on rational combinations of p_1^2, s and t , and a one- and (in the non-planar case) two-dimensional integrals with simple integrands. To do this, the method based on MB integrals described above was applied.

These and other master planar and non-planar double boxes with one leg off-shell were evaluated in [13] with the help of the method of differential equations [23]. The corresponding results are expressed through so-called two-dimensional harmonic polylogarithms which generalize harmonic polylogarithms introduced in [24].

A reduction procedure that provides the possibility to express any given Feynman integral to the master integrals was also developed in [13]. It is based on the observation that, when increasing the total dimension of the denominator and numerator in Feynman integrals associated with the given graph, the total number of IBP and LI equations grows faster⁴ than the number of independent Feynman integrals (labeled by the powers of propagators and powers of independent scalar products in the numerators). Therefore this system of equations sooner or later becomes overconstrained, and one obtains the possibility to perform a reduction to master integrals.

In fact, this strategy is quite general and can be applied to any Feynman graph. However its implementation for any concrete graph looks rather non-trivial. In particular, solving an overconstrained system of several thousand equations is a challenging technical task. Nevertheless, this method was successfully applied [26] to the Feynman integrals with one leg off-shell contributing to the process $e^+e^- \rightarrow 3\text{jets}$.

⁴This fact was first pointed out by Laporta and used in [25].

The massive on-shell master double box.

We now turn to the massive on-shell case, i.e. $p_i^2 = m^2$, $i = 1, 2, 3, 4$, and consider the general on-shell planar double box diagram of Fig. 4, i.e. with general irreducible

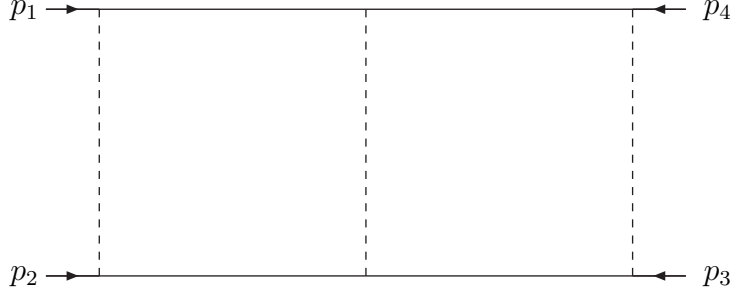


Figure 4: Planar double box diagram. Solid and dashed lines denote massive (with the mass m) and massless propagators, respectively.

numerator and powers of propagators. The irreducible numerator, the numbering of lines and the routing of the external momenta are chosen as in the massless case of (9) (see Fig. 3). This general double box Feynman integral takes the form

$$B(a_1, \dots, a_8; s, t, m^2; \epsilon) = \int \int \frac{d^d k d^d l}{(k^2 - m^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2 - m^2]^{a_3} [(k + p_1 + p_2 + p_3)^2]^{-a_8} [(l + p_1 + p_2)^2 - m^2]^{a_4} [(l + p_1 + p_2 + p_3)^2]^{a_5} (l^2 - m^2)^{a_6} [(k - l)^2]^{a_7}}. \quad (22)$$

To arrive at a MB representation (with, possibly, minimal number of MB integrations) one straightforwardly generalizes the above procedure for the massless case by introducing two extra MB integrations, when separating terms with m^2 after each of the integrations over the loop momenta, and after appropriate changes of variables one obtains the following sixfold MB representation of (22):

$$B(a_1, \dots, a_8; s, t, m^2; \epsilon) = \frac{(i\pi^{d/2})^2 (-1)^a}{\prod_{j=2,4,5,6,7} \Gamma(a_j) \Gamma(4 - a_{4567} - 2\epsilon) (-s)^{a-4+2\epsilon}} \times \frac{1}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} dw \prod_{j=1}^5 dz_j \left(\frac{m^2}{-s} \right)^{z_1+z_5} \left(\frac{t}{s} \right)^w \frac{\Gamma(a_2 + w) \Gamma(-w) \Gamma(z_2 + z_4) \Gamma(z_3 + z_4)}{\Gamma(a_1 + z_3 + z_4) \Gamma(a_3 + z_2 + z_4)} \times \frac{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3) \Gamma(a_{1238} - 2 + \epsilon + z_4 + z_5) \Gamma(a_7 + w - z_4)}{\Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - 2z_1 - z_2 - z_3)} \times \frac{\Gamma(a_{4567} - 2 + \epsilon + w + z_1 - z_4) \Gamma(a_8 - z_2 - z_3 - z_4) \Gamma(-w - z_2 - z_3 - z_4)}{\Gamma(4 - a_{1238} - 2\epsilon + w - z_4) \Gamma(a_8 - w - z_2 - z_3 - z_4)} \times \frac{\Gamma(a_5 + w + z_2 + z_3 + z_4) \Gamma(2 - a_{567} - \epsilon - w - z_1 - z_2)}{\Gamma(4 - a_{13} - 2a_{28} - 2\epsilon + z_2 + z_3 - 2z_5)}$$

$$\begin{aligned}
& \times \Gamma(2 - a_{457} - \epsilon - w - z_1 - z_3) \Gamma(2 - a_{128} - \epsilon + z_2 - z_5) \Gamma(2 - a_{238} - \epsilon + z_3 - z_5) \\
& \times \Gamma(4 - a_{46} - 2a_{57} - 2\epsilon - 2w - z_2 - z_3) \Gamma(-z_1) \Gamma(-z_5). \tag{23}
\end{aligned}$$

In the case of the master double box, we set $a_i = 1$ for $i = 1, 2, \dots, 7$ and $a_8 = 0$ and obtain a massive generalization of (11)

$$\begin{aligned}
B^{(0)}(s, t, m^2; \epsilon) & \equiv B(1, \dots, 1, 0; s, t, m^2; \epsilon) \\
& = -\frac{(i\pi^{d/2})^2}{\Gamma(-2\epsilon)(-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} dw \prod_{j=1}^5 dz_j \left(\frac{m^2}{-s}\right)^{z_1+z_5} \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
& \times \frac{\Gamma(2+\epsilon+w+z_1-z_4)\Gamma(-1-\epsilon-w-z_1-z_2)\Gamma(-1-\epsilon-w-z_1-z_3)\Gamma(-z_1)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)\Gamma(-2\epsilon+z_2+z_3-2z_5)} \\
& \times \frac{\Gamma(-\epsilon+z_2-z_5)\Gamma(-\epsilon+z_3-z_5)\Gamma(1+\epsilon+z_4+z_5)\Gamma(-z_5)\Gamma(-2\epsilon+z_2+z_3)}{\Gamma(-2-2\epsilon-2w-2z_1-z_2-z_3)} \\
& \times \Gamma(-2-2\epsilon-2w-z_2-z_3)\Gamma(1+w+z_2+z_3+z_4)\Gamma(z_2+z_4)\Gamma(z_3+z_4) \\
& \times \Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4). \tag{24}
\end{aligned}$$

The resolution of singularities in ϵ is performed [14] also in the standard way (see [5, 7, 11, 12] and the examples above) and reduces to shifting contours and taking residues. The final result takes the following form:

$$B^{(0)}(s, t, m^2; \epsilon) = -\frac{(i\pi^{d/2}e^{-\gamma_E\epsilon})^2 x^2}{s^2(-t)^{1+2\epsilon}} \left[\frac{b_2(x)}{\epsilon^2} + \frac{b_1(x)}{\epsilon} + b_{01}(x) + b_{02}(x, y) + O(\epsilon) \right], \tag{25}$$

where $x = 1/\sqrt{1-4m^2/s}$, $y = 1/\sqrt{1-4m^2/t}$,

$$b_2(x) = 2(m_x - p_x)^2, \tag{26}$$

$$\begin{aligned}
b_1(x) & = -8 \left[\text{Li}_3\left(\frac{1-x}{2}\right) + \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{-2x}{1-x}\right) + \text{Li}_3\left(\frac{2x}{1+x}\right) \right] \\
& + 4(m_x - p_x) \left[\text{Li}_2\left(\frac{1-x}{2}\right) - \text{Li}_2\left(\frac{-2x}{1-x}\right) \right] - (4/3)m_x^3 + 4m_x^2 p_x \\
& - 6m_x p_x^2 + (2/3)p_x^3 + 4l_2(m_x p_x + p_x^2) - 2l_2^2(m_x + 3p_x) \\
& - (\pi^2/3)(4l_2 - m_x - 3p_x) + (8/3)l_2^3 + 14\zeta_3, \tag{27}
\end{aligned}$$

$$\begin{aligned}
b_{01}(x) & = -8(m_x - p_x) \left[\text{Li}_3(x) - \text{Li}_3(-x) - \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{1-x}{2}\right) \right. \\
& \left. - \text{Li}_3\left(\frac{2x}{1+x}\right) + \text{Li}_3\left(\frac{-2x}{1-x}\right) \right] + 4 \left[\text{Li}_2(x)^2 + \text{Li}_2(-x)^2 + 4\text{Li}_2\left(\frac{1-x}{2}\right)^2 \right] \\
& - 8\text{Li}_2(x)\text{Li}_2(-x) + 16\text{Li}_2\left(\frac{1-x}{2}\right)(\text{Li}_2(x) - \text{Li}_2(-x)) \\
& - (4/3)[\pi^2 - 6l_2^2 + 3m_x^2 + 6m_x(2l_2 - 2l_x - p_x) + 12l_x p_x - 3p_x^2](\text{Li}_2(x) - \text{Li}_2(-x)) \\
& - (8/3)[\pi^2 - 6l_2^2 + 6l_x p_x - 6m_x(l_x + p_x - 2l_2)]\text{Li}_2\left(\frac{1-x}{2}\right)
\end{aligned}$$

$$\begin{aligned}
& +8(m_x - p_x) \left[(p_x - m_x + 2l_2) \text{Li}_2 \left(\frac{2x}{1+x} \right) + 2(l_x - m_x + l_2) \text{Li}_2 \left(\frac{-2x}{1-x} \right) \right] \\
& -8(m_x - p_x)(2l_x - p_x - 5m_x + 4l_2)(-m_x p_x + l_2(m_x + p_x) - l_2^2 + \pi^2/6) \\
& -(20/3)m_x^4 + (164/3)m_x^3 p_x - 40m_x^2 p_x^2 - (4/3)m_x p_x^3 - (8/3)p_x^4 \\
& +8m_x l_x (m_x^2 - 3m_x p_x + 2p_x^2) \\
& -4l_2(7m_x^3 + 21m_x^2 p_x - 4m_x l_x p_x - 23m_x p_x^2 + 4l_x p_x^2 - p_x^3) \\
& -\pi^2((17/3)m_x^2 - (4/3)m_x l_x - 2m_x p_x + (4/3)l_x p_x - (7/3)p_x^2) \\
& +l_2^2(84m_x^2 - 8m_x l_x - 16m_x p_x + 8l_x p_x - 44p_x^2) \\
& -(8/3)l_2(6l_2^2 - \pi^2)(3m_x - 2p_x) - (4/3)\pi^2 l_2^2 + 4l_2^4 + \pi^4/9, \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
b_{02}(x, y) = & 2(p_x - m_x) \left\{ 4 \left[\text{Li}_3 \left(\frac{1-x}{2} \right) - \text{Li}_3 \left(\frac{1+x}{2} \right) + \text{Li}_3 \left(\frac{(1-x)y}{1-xy} \right) \right. \right. \\
& \left. \left. - \text{Li}_3 \left(\frac{-(1+x)y}{1-xy} \right) + \text{Li}_3 \left(\frac{-(1-x)y}{1+xy} \right) - \text{Li}_3 \left(\frac{(1+x)y}{1+xy} \right) \right] \right. \\
& + 2 \left[\text{Li}_3 \left(\frac{(1+x)(1-y)}{2(1-xy)} \right) - \text{Li}_3 \left(\frac{(1-x)(1+y)}{2(1-xy)} \right) \right. \\
& \left. \left. - \text{Li}_3 \left(\frac{(1-x)(1-y)}{2(1+xy)} \right) + \text{Li}_3 \left(\frac{(1+x)(1+y)}{2(1+xy)} \right) \right] \right. \\
& + 2(m_y + p_y - m_{xy} - p_{xy}) \left[2\text{Li}_2(x) - 2\text{Li}_2(-x) + \text{Li}_2 \left(\frac{-2x}{1-x} \right) - \text{Li}_2 \left(\frac{2x}{1+x} \right) \right] \\
& + 4(m_{xy} - p_{xy})(\text{Li}_2(-y) - \text{Li}_2(y)) - 4(m_x + p_x - 2l_2) \text{Li}_2 \left(\frac{1-x}{2} \right) \\
& - 4(m_{xy} - p_{xy}) \text{Li}_2 \left(\frac{1-y}{2} \right) - 4(m_x + l_y - m_{xy}) \text{Li}_2 \left(\frac{(1-x)y}{1-xy} \right) \\
& + 4(p_x + l_y - m_{xy}) \text{Li}_2 \left(\frac{-(1+x)y}{1-xy} \right) - 4(m_x + l_y - p_{xy}) \text{Li}_2 \left(\frac{-(1-x)y}{1+xy} \right) \\
& + 4(p_x + l_y - p_{xy}) \text{Li}_2 \left(\frac{(1+x)y}{1+xy} \right) \\
& + 2(m_x + p_x + m_y + p_y - 2m_{xy} - 2l_2) \text{Li}_2 \left(\frac{(1-x)(1+y)}{2(1-xy)} \right) \\
& + 2(m_x + p_x + m_y + p_y - 2p_{xy} - 2l_2) \text{Li}_2 \left(\frac{(1-x)(1-y)}{2(1+xy)} \right) \\
& + 2p_x^2(m_y + p_y - m_{xy} - p_{xy}) + 2p_x(2(m_y l_y + m_y p_y + l_y p_y) \\
& + m_{xy}(-m_y - 2l_y - 3p_y + 3m_{xy}) + p_{xy}(-3m_y - 2l_y - p_y + 3p_{xy})) \\
& + 2m_x(2p_x + m_y - 2l_y + p_y)(m_y + p_y - m_{xy} - p_{xy}) - p_y^2(m_{xy} + p_{xy}) \\
& + 2p_y(2m_{xy}^2 + p_{xy}^2) + m_y^2(2p_y - m_{xy} - p_{xy})
\end{aligned}$$

$$\begin{aligned}
& +2m_y(p_y^2 + m_{xy}^2 + 2p_{xy}^2 - p_y(3m_{xy} + p_{xy})) - 2(m_{xy}^3 + p_{xy}^3) \\
& + 2l_2((4m_y + 4p_y - 3m_{xy})m_{xy} + (2m_y + 2p_y - 3p_{xy})p_{xy} \\
& - 2(p_x + 2m_x)(m_y + p_y - m_{xy} - p_{xy}) - m_y^2 - 4m_y p_y - p_y^2) \\
& + 2l_2^2(3(m_y + p_y) - 2(2m_{xy} + p_{xy})) \\
& - (\pi^2/3)(m_y + p_y - 8m_{xy} + 6p_{xy}) \} . \tag{29}
\end{aligned}$$

Here the following abbreviations are used: $\zeta_3 = \zeta(3)$, $l_z = \ln z$ for $z = x, y, 2$, $p_z = \ln(1 + z)$ and $m_z = \ln(1 - z)$ for $z = x, y, xy$.

The result (25)–(29) is in agreement with the leading power behaviour in the (Sudakov) limit of the fixed-angle scattering, $m^2 \ll |s|, |t|$. This asymptotics is obtained by use of the strategy of expansion by regions [17, 18, 19]. The structure of regions is very rich. The following family of seventeen regions participates here:

$$\begin{aligned}
& (h-h), (1c-h), \dots, (4c-h), (1c-1c), \dots, (4c-4c), \\
& (1c-3c), (2c-4c), (1c-4c), (2c-3c), \\
& (1uc-2c), (2uc-1c), (3uc-4c), (4uc-3c).
\end{aligned}$$

Here h denotes hard, c – collinear and uc – ultracollinear regions for the two loop momenta. (See [18] and Chapter 8 of [19] for definitions of these regions.) In particular, the (h–h) contribution is nothing but the massless on-shell double box (19). Evaluating and summing up all the contributions, we obtain [14]

$$\begin{aligned}
B^{(0)}(s, t, m^2; \epsilon) = & -\frac{(i\pi^{d/2}e^{-\gamma_E\epsilon})^2}{s^2(-t)^{1+2\epsilon}} \left\{ 2\frac{L^2}{\epsilon^2} - \left[(2/3)L^3 + (\pi^2/3)L + 2\zeta_3 \right] \frac{1}{\epsilon} \right. \\
& - (2/3)L^4 + 2\ln(t/s)L^3 - 2(\ln^2(t/s) + 4\pi^2/3)L^2 \\
& + \left[4\text{Li}_3(-t/s) - 4\ln(t/s)\text{Li}_2(-t/s) + (2/3)\ln^3(t/s) - 2\ln(1+t/s)\ln^2(t/s) \right. \\
& \left. \left. + (8\pi^2/3)\ln(t/s) - 2\pi^2\ln(1+t/s) + 10\zeta_3 \right] L + \pi^4/36 \right\} + O(m^2L^3, \epsilon), \tag{30}
\end{aligned}$$

where $L = \ln(-m^2/s)$. This asymptotic behaviour is reproduced when one starts from result (25)–(29).

It is interesting to note that, in the above result, there are no so-called two-dimensional harmonic polylogarithms [24] which have turned out to be adequate functions to express results for the double boxes with one leg off-shell [13].

Perspectives.

It is believed that sooner or later we shall achieve the limit in the process of analytical evaluation of Feynman integrals so that we shall be forced to proceed only numerically. (See, e.g., the first paper of [27] where this point of view has been emphasized.) The progress in the field of numerical evaluation of Feynman integrals was rather visible last years. Several new powerful numerical methods have been developed — see, e.g., [28, 29, 30, 27]. They have been applied in practice and, at the same time, served for crucial checks of analytical results. For example, the method

of [28] applicable to Feynman integrals with severe ultraviolet, infrared and collinear divergences was successfully applied to check a lot of results for the double-box master integrals presented and/or discussed above.

However the dramatic progress in the field of analytical evaluation of Feynman integrals shows that we have not yet exhausted our abilities in this direction. Let us take the problem of the analytical evaluation of the on-shell double-box diagrams with $p_i^2 = m^2$, $i = 1, 2, 3, 4$ as an example. The calculational experience, in particular obtained when evaluating four-point diagrams, tell us that if such master integrals can be evaluated, the problem can be also completely solved, after evaluating other master integrals and constructing a recursive procedure that expresses any given Feynman integral with general numerators and integer powers of propagators through the master integrals. Therefore the explicit analytical result (25)–(29) can be considered as a kind of existence theorem, in the sense that it strongly indicates the possibility to analytically compute a general Feynman integral in this class (and apply these results to various scattering processes in two loops without putting masses to zero). To calculate the master integrals one can apply the technique of MB integration described above. To construct appropriate recursive algorithms both in the planar and non-planar case one can use recently developed methods based on shifting dimension [20] and differential equations [13] as well a method based on non-recursive solutions of recurrence relations [31] (successfully applied in practice [32]).

One can also hope that new analytical results can be obtained for many other classes of Feynman integrals depending on two and three scales. In particular, the analytical evaluation of any two-loop two-scale Feynman integral with two, three and four legs looks quite possible.

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